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Stability and BPS branes

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Abstract: We define the concept of Π-stability, a generalization of μ-stability of vector bundles, and argue that it characterizes $\mathcal{N}=1$ supersymmetric brane configurations and BPS states in very general string theory compactifications with $\mathcal{N}=2$ supersymmetry in four dimensions.

Keywords: D-branes, Superstring Vacua.

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1. Introduction

Compactifications of the heterotic string or type-II superstring theory with D-branes include a choice of gauge field configuration. Characterizing the possibilities and analyzing their physics is a central problem in this subject.

The most intensively studied case is a gauge field on a Calabi-Yau manifold preserving $\mathcal{N} = 1$ supersymmetry in four dimensions. As is well-known, solutions of the Yang-Mills equations preserving this supersymmetry correspond by the work of Donaldson, Uhlenbeck and Yau to holomorphic vector bundles satisfying the condition of μ-stability, a quasi-topological condition depending on the Kähler class of the Calabi-Yau.

In general, quantities which depend on the Kähler class are modified in string theory by world-sheet instanton corrections. In the case of heterotic strings, space-time instanton corrections can also enter. The true picture of this moduli space can in some cases be obtained by duality; mirror symmetry in type-II compactification and type II-heterotic duality in heterotic compactification. These corrections can drastically alter the large volume and classical picture, and this strongly suggests that the μ-stability condition must be significantly modified as well.

Based on recent work on BPS D-brane configurations in Calabi-Yau compactification, we propose a generalization of the μ-stability condition which takes these corrections into account. The dependence on the Kähler class is replaced by dependence on the periods Π

of the Calabi-Yau and thus we call it "Π-stability." The usual definitions of Calabi-Yau periods and $\mathcal{N}=2$ central charges must be generalized slightly (in a way already suggested by mathematicians) to make the definition, as we will also explain.

Π-stability is a precisely defined condition and can be studied using the same mathematical techniques as μ-stability; we will consider the noncompact Calabi-Yau $\mathcal{O}_{\mathbb{P}^2}(-3)$ in detail in an upcoming work [1]. In the present work we will state the ideas and assumptions which lead us to this proposal and check that it is compatible with the known physics of BPS branes and marginal stability in solvable examples; the large volume limit, the orbifold limit and the large complex structure limit, in which it is related to a condition governing stability of special lagrangian manifolds formulated by Joyce [2].

2. The proposal

Let $\mathcal{M}_c$ and $\mathcal{M}_k$ be the complex structure and complexified Kähler moduli spaces of a Calabi-Yau M in string theory, with complex dimension $b_{2,1}$ and $b_{1,1}$. $\mathcal{M}_k$ is best defined (when possible) as the complex structure moduli space of a mirror manifold W. Let z^i be local coordinates on $\mathcal{M}_k$ and $\Pi_a \in \mathbb{C}^{2b_{1,1}+2}$ be a vector of periods as defined in special geometry (e.g. when a mirror exists, the periods $\int \Omega$ of the holomorphic three-form of the mirror). These are defined up to overall normalization; we choose a particular (but arbitrary) normalization at each point in $\mathcal{M}_k$.

Let E be a holomorphic cycle carrying a vector bundle, or some generalization of this idea appropriate to string theory. The proposal rests on the idea that these can be defined knowing only the complex structure of M. We require a category of these and an idea of homomorphism. E' is a subbundle of E if $E' \neq E$ and there exists a holomorphic embedding of E' in E or in other words an injective holomorphic map from sections of E' to E. More generally, E' is a subobject of E if there is a monomorphism (an injective homomorphism) in $\mathrm{Hom}(E', E)$.

Let $Q(E)$ be the charge of E or appropriate generalization, defined so that the central charge of a brane associated with E is $Z = Q(E)\cdot\Pi$. Clearly the precise definition of $Q(E)$ depends on our basis for Π. In the A picture (special lagrangians on W), $Z = \int_\Sigma \Omega$ for a brane wrapped on the cycle Σ and we are just choosing a basis for $H^3(W)$. In the B picture, a definition with strong motivations from D-brane physics and the mathematics of K-theory takes $Q(E) = \mathrm{ch}(E)\sqrt{\hat{A}}$ where $\mathrm{ch}(E)$ is the Chern character and $\sqrt{\hat{A}}$ a topological invariant of the CY. By taking a different basis for the periods, one could also work in conventions where $Q(E) = \mathrm{ch}(E)$, which tend to be more convenient for comparisons with the mathematical literature on vector bundles. In any case, we want to emphasize that the concepts entering our definitions are independent of convention.

We next define the "grade" $\varphi(E)$ of a BPS brane at a point in moduli space with periods Π to be

$$\begin{aligned} \varphi(E) &= \frac{1}{\pi}\arg Z(E) \\ &= \frac{1}{\pi}\mathrm{Im}\log Z(E). \end{aligned} \tag{2.1}$$

One might also write $\varphi(E;\Pi)$ to make the dependence on the periods explicit. The terminology is intended to be an analog both of the slope of a vector bundle and of the grading in a derived category. Note that the grade will ultimately be defined to take values in $\mathbb{R}$, not $[0,2)$. Given the grading at some point in moduli space, we will define them elsewhere by analytic continuation of $Z(E)$. Clearly the well-definedness of this will require some discussion, which we will make below.

We now define E to be Π-stable at a point in moduli space with periods Π if, for every subobject $E' \subset E$, we have

$$\varphi(E') < \varphi(E)\,. \tag{2.2}$$

We then conjecture that the BPS branes in the theory (for bulk moduli Π) are the Π-stable objects with unbroken gauge symmetry U(1).

When (2.2) degenerates to equality, it is clear that a decay of E to products including E' would be physically allowed. The additional physical information in the statement of Π-stability is the statements that certain decays $E \to E' + \ldots$ are not possible despite the degeneration of (2.2) (namely, those for which E' is not a subobject), that the bound state E will exist on a specific side of this line and not on the other, that the other cases $\varphi(E) - \varphi(E') \in \mathbb{Z}$ do not lead to decays, and that all objects not destabilized by particular subobjects actually exist. We proceed to check these points in various limits of the theory.

3. The large volume limit

In the large volume limit of $\mathcal{M}_k$ Π-stability (2.2) reduces to μ-stability. In this limit, the periods can be associated with the $2k$-cycles of M, and in terms of the triple intersection form

$$c_{ijk} = \int_M \omega_i \wedge \omega_j \wedge \omega_k$$

we have (up to lower order corrections)[1]

$$\begin{aligned}
\Pi_6 &= \frac{1}{6} c_{ijk} t^i t^j t^k \\
\Pi_4^i &= -\frac{1}{2} c_{ijk} t^j t^k \\
\Pi_2^i &= t^i \equiv B^i + iV^i \\
\Pi_0 &= -1
\end{aligned}$$

with $B = \sum \omega_i \,\mathrm{Re}\, t^i$ and $J = \sum \omega_i \,\mathrm{Im}\, t^i$. The leading terms ($|B| \ll V$) in (2.1) then take the form

$$\begin{aligned}
\varphi(E) &= \frac{1}{\pi} \mathrm{Im} \log \Pi_6 + \mathrm{Im} \frac{1}{2\pi \Pi_6 \mathrm{rank}\ E} \int J \wedge J \wedge \mathrm{c}_1(E) \\
&\sim \frac{3}{2} + \frac{3}{\pi V} \frac{1}{\mathrm{rank}\ E} \int J \wedge J \wedge \mathrm{c}_1(E) + \cdots\,.
\end{aligned}$$

[1]The conventions are the ones in which $Q_{2p} = \mathrm{ch}_{3-p}(E)$.

The leading nonconstant term for $V \gg 1$ and $V \gg |B|$ is proportional to the slope, so in this limit Π-stability reduces to μ-stability [3].

Some features of the world-volume physics of μ-stability have been discussed by Sharpe [4]. For $b^{1,1} > 1$, one can have "walls" in Kähler moduli space on which the μ-stability of bundles changes. These have been much studied for complex surfaces for application to Donaldson theory. Physically, a wall-crossing leading to decay of a bound state corresponds to an enhancement of gauge symmetry on the wall (where the bundle is semistable) followed by D-term supersymmetry breaking, in the simplest case governed by the potential

$$V = (|\phi|^2 - \zeta)^2 . \tag{3.1}$$

The same relation to the D-terms will hold for Π-stability.

One can get a first indication of why we will need to extend the grading beyond the region $[0, 2)$ by dropping the condition $|B| \ll V$. Since string theory only depends on the gauge-invariant combination $B - F$, the situation with arbitrary values of B can be related to that for $|B| \ll V$ by considering branes with non-zero $c_1 = F$, i.e. tensoring our bundles with appropriate line bundles. Once we start to consider c_1 of order V (in string units), the grading will leave the region $[0, 2)$: for example, using line bundles we can get $\frac{1}{\pi} \operatorname{Im} \log(-F + iV)^3$ which takes values in $[0, 3)$.

Although one might wonder if the stability of a brane with such large values of the flux changes from the usual large volume idea, we have no evidence for this. Actually, if we do not extend the grading to a nonperiodic variable, the definition (2.2) of stability is not sensible.

According to our definition of subobject, a single six-brane carrying a line bundle with $\int J \wedge J \wedge c_1 = n$ (let us call it $O(n)$) will have infinitely many subobjects, namely the branes $O(m)$ with $m < n$. Although these are not considered subobjects for the usual definition of μ-stability (they are never relevant for this definition anyways), they can become relevant away from the large volume limit, and it is not natural to drop them. If we do not extend the grading, these lead to many nonsensical predictions; in particular decays of branes into heavier constituents.

Requiring that in the $c_1 \to \infty$ limit, the grading goes over to that for the D0-brane suggests setting the gradings for pure (trivial bundle) $2p$-branes to be $\varphi(\Sigma_{2p}) = 3 - p/2$, in the conventions of this section. (Any choice for the origin $\varphi = 0$ is of course a convention.)

We will give further arguments below for why extending the grading is sensible and correct in string theory.

4. Considerations from world-volume effective theory

As we go away from the large volume limit, our primary tools will be the constraints of $\mathcal{N} = 1$ supersymmetry on the world-volume effective theory, and the "decoupling" statement of [5], that superpotential and D terms only depend respectively on the complex and Kähler moduli for B branes and the mirror for A branes. This statement will be further justified elsewhere.

Clearly the configurations of a single BPS brane on a CY can be described by a $d = 4$, $\mathcal{N} = 1$ effective world-volume theory. We will also consider non-BPS combinations of branes and non-BPS branes which can be described by non-supersymmetric vacua of such a theory, obtained by combining the theories of the various BPS constituents and adding degrees of freedom corresponding to open strings stretched between these branes.

It is not strictly true that all BPS bound states are described by supersymmetric vacua of the resulting world-volume theory. This condition is too restrictive as it ignores the possibility that a different $\mathcal{N} = 1$ supersymmetry is unbroken, given by a combination of the linearly realized $\mathcal{N} = 1$ supersymmetry and an inhomogeneous $\mathcal{N} = 1$ symmetry present in all theories containing a decoupled U(1) (such as D-brane bound states). This latter is simply the shift of the decoupled gaugino $\delta\chi = \epsilon'$.

Such supersymmetry preserving vacua are characterized by having a non-zero potential which comes entirely from an overall constant shift of the U(1) D terms (i.e. $D_i = D_j$ for every U(1) factor). Other supersymmetry breaking vacua, in particular those in which the F terms are non-zero, correspond to non-BPS bound states. The states found by Sen in K3 compactification [6] are an example which can be understood this way.[2]

A way to distinguish the non-BPS situation from the BPS vacuum preserving a different $\mathcal{N} = 1$ supersymmetry is to note that D terms do not give masses to fermions, while F terms do. Thus the possibility of two BPS branes combining to a BPS bound state is signalled by a massless Ramond open string fermion living in a chiral multiplet (it is massless when gauge symmetry is unbroken; i.e. at the unstable maximum of the D term potential). We will use these characterizations of BPS ground states below.

We finally remark on the relation between "non-BPS branes" and supersymmetry breaking in physical string compactifications. First, to get supersymmetry breaking one should break all of the $\mathcal{N} = 1$ supersymmetries, so indeed one is looking for "non-BPS" branes with breaking by F and D terms together. One must furthermore ensure breaking for all values of the bulk moduli, which has not been accomplished in examples considered so far.

5. D-branes and homological algebra

To study D-branes and their stability at arbitrary points in moduli space, we need to describe them as objects which allow a study of their moduli space and their subobjects (potential decay products). A natural framework for this seems to be homological algebra [7, 8], and in particular the language of extension groups, $\mathrm{Ext}^p(V, W)$. We will try now to give some physical intuition for the role of $\mathrm{Ext}^p(V, W)$.

For bundles, $\mathrm{Ext}^p(V, W)$ is the same as $H^p(M, V^* \otimes W)$. Recall that one physical appearance of the complex cohomology groups is that they count fermion zero modes. As is well-known, holomorphic p-forms on a Calabi-Yau manifold are directly related to spinors with chirality determined by the parity of p. An element of $H^p(M, V^* \otimes W)$ will thus correspond to a zero mode of the Dirac operator coupled to the bundle $V^* \otimes W$.

[2]This point was developed in a discussion with A. Sen.

In the general discussion of BPS branes as quantum objects [9] and in $(0,2)$ sigma models of heterotic string compactification [10], sheaves can sometimes be used instead of bundles, and the generalization to Ext has been used in this context.

Another context in which the Ext groups are useful turns out to be quiver gauge theories. Indeed these provide very elementary examples, which will be discussed at length in [1]. Let us give the briefest introduction here, to make some necessary points.

We recall that a quiver is a directed graph with vertices $v \in V$ and arrows $a \in A$ from vertices ta to ha; an associated gauge theory is labelled by a dimension (we also use the terms weight or charge) vector n. It has gauge group $\prod_v U(n_v)$, matter content $R^a_{i,\bar{j}}$ transforming in $(n_{ta}, \bar{n}_{ha})$ and a superpotential.

A moduli space of solutions to the superpotential constraints $W' = 0$ is a complex variety, and the set of these for various n_i provides another example of the type of category of holomorphic objects we have in mind. Following the quiver literature, we will refer to points in these moduli spaces as "representations" of the quiver.

A homomorphism between two representations R and S is defined as a set of linear maps $\phi_v : R_v \mapsto S_v$ for each vertex v satisfying $S^a\phi_{ta} = \phi_{ha}R^a$ for each arrow a. We say that R is a subrepresentation of S if there is an injective homomorphism from R to S.

For $p = 0$, we have $\mathrm{Ext}^0(V, X) = \mathrm{Hom}(V, X)$. The definition we gave of subobject in section 2 makes sense in any abelian category: we say that V is a subobject of X is there is an injective homomorphism from V to X. We will see in section 7 that subobjects play the same role in quiver gauge theory that they did in the discussion of bundles.

We note that the relation of subobject is not necessarily determined by the charges of the two objects; two objects of the same charge might differ in the charges of subobjects they admit. A common situation is that all "generic" objects of a given charge admit the same subobjects, while certain degenerate objects (for example semisimple ones) admit more. If so, we can (a bit loosely) talk about one weight vector $n(E')$ being a subobject of another weight vector $n(E)$. This means that a generic representation E of weight $n(E)$ will have a subrepresentation E' of weight $n(E')$.

Moving to $p = 1$ and Ext (one sometimes leaves off the superscript 1), it is well known that elements of $H^1(M, \mathrm{End}V)$ correspond to deformations of the complex structure of V. A related statement which is relevant for bound state problems is that $H^1(M, V^* \otimes W)$ corresponds to possible deformations of the direct sum bundle $V \oplus W$. Generalizing these observations to arbitrary points in moduli space, one criterion we might apply to find out if V and W can form a bound state is to ask if $\mathrm{Ext}(V, W) \oplus \mathrm{Ext}(W, V) \neq 0$.

In general, an extension X is associated with an exact sequence

$$0 \longrightarrow W \xrightarrow{\phi} X \longrightarrow V \longrightarrow 0$$

which does not split: $X \neq W \oplus V$. This is also the definition of extension for quiver representations, and can be used to define $\mathrm{Ext}^1(V, W)$ in this context.

The second arrow in this sequence, representing $\phi \in \mathrm{Hom}(W, X)$, gives us a way to think of a homomorphism as associated with a particular way to form a bound state. Another picture is that these are "potential gauge symmetries," which can become un-

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broken when the bound state becomes marginally stable. This will be signalled by the appearance of a $\mathrm{Hom}(X,W)$ which (if W and X are not isomorphic) implies that X is no longer simple.

A primary tool for determining the dimensions of these groups is the Grothendieck-Riemann-Roch theorem, which can be thought of as a special case of the index theorem for the Dirac operator applicable for holomorphic bundles. This generalizes to sheaves and quivers, and allows evaluating

$$\chi(V,W) = \sum_i (-1)^i \dim \mathrm{Ext}^i(V,W) \tag{5.1}$$

in terms of the Chern classes of V, W and M. Although this does not determine any of the individual dimensions directly, given appropriate further assumptions one might use it to make statements such as $\dim \mathrm{Hom} = 0$ implies $\chi \leq 0$.

For bundles on a Calabi-Yau, $\chi(E',E)$ is mirror to the intersection number $I(E',E)$ between three-branes, and is antisymmetric. For bundles on a divisor of a CY, the two are related as

$$I(E',E) = \chi(E',E) - \chi(E,E')\,. \tag{5.2}$$

Whereas the intersection number counts all fermionic massless strings between E' and E, in this case χ distinguishes some of these and gives more information.

As an elementary example, let us consider the quiver with two nodes and three arrows between them (and no superpotential). Its representations correspond to arbitrary configurations of three chiral multiplets which transform as $(\bar{n}_1, n_2)$ under the gauge group $\mathrm{U}(n_1) \times \mathrm{U}(n_2)$. The analog of (5.1) in this case is [11, 1]:

$$\chi(E',E) = \dim \mathrm{Hom}(E',E) - \dim \mathrm{Ext}(E',E) = n'_1 n_1 + n'_2 n_2 - 3n'_1 n_2\,.$$

This theory also turns out to describe the moduli spaces of certain bundles on $\mathbb{P}^2$ [1] and (5.2) is the corresponding intersection form on the local mirror to $\mathbb{C}^3/\mathbb{Z}^3$.

The simplest example of a subobject is $E' = (0\ 1)$ which is a subobject of any quiver representation $(n_1\ n_2)$ with $n_2 > 0$ (the relation defining the homomorphism degenerates). As a more subtle example, we give the pair $E' = (1\ 0)$ and $E = (3\ 1)$ for which $\dim \mathrm{Hom}(E',E) = 0$, $\dim \mathrm{Hom}(E,E') = 3$, and the Ext groups are zero. This allows us to illustrate several points: *i*) since $\mathrm{Hom}(E',E) = 0$, E' can not be a subobject of E; *ii*) although $\mathrm{Hom}(E,E') \neq 0$, E is clearly not a subobject of E'; *iii*) the vanishing of the Ext groups corresponds to the fact that (4 1) is not a bound state (all such configurations have unbroken gauge symmetry); *iv*) the intersection number $I(E',E) = -3$ by itself is not enough information to see this.

We gave two examples of categories which appear in string theory, but an important question is whether some category of holomorphic objects can describe all B branes (in some background CY) over all of Kähler moduli space. As we discuss in the conclusions, the derived category of coherent sheaves is a natural candidate.

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6. Marginal stability and special lagrangian geometry

For a given brane (bundle) E and subbrane E', the inequality in condition (2.2) will degenerate to equality on walls of real codimension one in $\mathcal{M}_k$. These are the familiar "lines of marginal stability" in $\mathcal{N} = 2$, $d = 4$ supersymmetric theories which physically are the only lines on which the condition for E to exist as a BPS brane could change. We now argue that the bound state E will exist on the side of the line predicted by (2.2) and not on the other.

The mirror interpretation of these processes involves joining and splitting of special lagrangian submanifolds on the mirror manifold W, as was studied by Joyce [2]. As one varies the complex structure of W, it is possible for a pair of special lagrangian manifolds of homology class $[\Sigma_1]$ and $[\Sigma_2]$ to intercommute producing a single manifold of class $[\Sigma_1] + [\Sigma_2]$. Conversely, a single brane can become unstable to split into a pair.

Joyce found a condition ([2], section 7) which applies to the local neighborhood of an intersection and to branes with small differences in the grade (in our terminology), and predicts on which side of the marginal stability line the bound state will be stable. This condition agrees precisely with (2.2) if we require that the intersection number $E \cdot E' > 0$. If it is negative, the role of the two branes is exchanged.

In string theory, this process can also be understood as a stretched open string between the two three-branes becoming tachyonic. Approximating the neighborhood of the intersection as flat space, there are six complex fields describing light strings stretched between D3-branes; their squared masses are linear in the angles of rotation. These masses can be described as a combination of superpotential and D-term masses in an effective $\mathcal{N} = 1$ field theory, and crossing the line of marginal stability changes the sign of the D-term mass, as was pointed out in [12].

As we argued in section 4, only the solutions with vacuum energy coming entirely from D terms can correspond to BPS bound states, and these can be identified by the presence of massless fermions. This means that the geometric condition $E \cdot E' > 0$ for A branes to intersect is not actually the condition which governs BPS decay. The correct condition instead, as suggested in section 5, is that $\mathrm{Hom}(E', E) \neq 0$ and $\mathrm{Hom}(E, E') = 0$.

This is a stronger condition than non-zero intersection number and the idea that a non-zero intersection number implies the existence of such a decay is contradicted in numerous examples on the B side. Indeed in orbifold theories one can realize the counterexample given in section 5. A further complication with predicting decays on the A side is that the superpotential can also have world-sheet instanton corrections [13], which could lift the degrees of freedom responsible for the decay.

Deriving holomorphic properties from the special lagrangian picture looks difficult at present. In practical applications of mirror symmetry, one is usually better off doing computations on the side which does not receive stringy corrections. For the holomorphic structure and specifically the computation of $\dim \mathrm{Hom}(E', E)$ this means the B side.

Given the existence of the homomorphism (so the decay can happen), we are basically asserting that the special lagrangian picture correctly predicts the direction of the decay. This seems almost beyond doubt in the large volume limit on W or equivalently the large complex structure limit in $\mathcal{M}_c$. Given our claim that $\mathcal{M}_c$ and $\mathcal{M}_k$ are effectively decoupled for the question of stability, this result is very strong.

7. The orbifold limit

A very different region of moduli space which can be treated exactly is the orbifold $\mathbb{C}^3/\Gamma$ with $\Gamma \subset SU(3)$ and the non-compact Calabi-Yaus obtained by substringy resolution of the singularity. As explained in [14–16], very general D-branes on these spaces are described by quiver gauge theories, with RR charges mapped into the ranks of the gauge groups. We now ask whether a BPS state exists with a particular charge vector. In this formalism it will be a bound state of fractional branes, and we need to know whether the associated gauge theory admits an $\mathcal{N} = 1$ supersymmetric vacuum (in the sense of section 4) which breaks the gauge symmetry to U(1).

The question of marginal stability in this context becomes the following. Let us imagine we have some solutions to the superpotential constraints (quiver representations): for what values of the Kähler moduli do they correspond to BPS states?

The dependence of the gauge theory potential on the Kähler moduli of the background CY is through Fayet-Iliopoulos terms; in other words the moment maps for the U(1)'s. Denote these as θ_i; they will satisfy $\sum_i \theta_i = 0$. For a cyclic orbifold $\mathbb{C}^3/\mathbb{Z}_n$ they satisfy no other relations; the real dimension of $\mathcal{M}_k$ equals the number of remaining FI terms.

The question of whether such a moduli space actually contains a supersymmetric vacuum can be answered using the work of King on stability of quiver representations [17]. This will be true if and only if the representation R is a direct sum of θ-stable representations. A θ-stable representation R is one for which

$$\sum_v \theta_v n_v(R) = 0 \tag{7.1}$$

and for every subrepresentation R' we have

$$\sum_v \theta_v n_v(R') > 0\,. \tag{7.2}$$

The condition (7.1) simply follows by taking traces of the D-flatness conditions. The condition (7.2) is proven by the techniques of geometric invariant theory. In general one finds a solution of the D-flatness conditions by minimizing the potential in a complexified gauge orbit. A subrepresentation violating (7.2) exists if and only if a one-parameter subgroup of a certain central extension (depending on θ) of the complexified gauge group with a limit point exists; in this case the minimum of the potential is at the limit point and off of the original complexified gauge orbit.

A very simple example illustrating (7.2) is the quiver of section 5. The fact that (0 1) is a subobject of any $(n_1\ n_2)$ with $n_2 > 0$ implies the (obvious) condition that non-trivial supersymmetric vacua exist only if $\theta_2 > 0$. More complicated quivers are required to illustrate the possibility of more complicated decays [1].

As we discussed earlier, the most general vacuum corresponding to a BPS state can have a constant non-zero potential arising from D terms. Given the physical FI parameters ζ_i, such a vacuum can be obtained by using a solution to the D-flatness conditions for a different set of 'FI terms' θ_i. Explicitly, we have

$$\begin{aligned} V &= \sum_i \mathrm{tr}(D_i - \theta_i + \theta_i - \zeta_i)^2 \\ &= \sum_i n_i(\theta_i - \zeta_i)^2 \,. \end{aligned}$$

The supersymmetric minimum will use the choice of θ satisfying (7.1) which minimizes the total energy.

Very near the orbifold point, we can use a quadratic approximation to the potential and kinetic term. The θ minimizing the potential is then

$$\theta = \zeta - \frac{\zeta \cdot n}{e \cdot n} e$$

where e is the vector with all components 1. The condition for θ-stability is then

$$\frac{\zeta \cdot n'}{e \cdot n'} > \frac{\zeta \cdot n}{e \cdot n} \,. \tag{7.3}$$

To compare this result with the prediction of Π-stability, we need an expression for the FI terms in terms of the periods Π. The orbifold points have an enhanced discrete symmetry and this allows us to write $\Pi_k \sim \frac{1}{n} - \sum z_m e^{2\pi i m k/n}$ at linear order. On the other hand, the FI terms are cyclically permuted under the symmetry; there exists a basis (the twist fields at the orbifold point) in which[3] $\zeta_m = -\,\mathrm{Im}\,\Pi_m$. Substituting this into (2.2) reproduces (7.3).

Thus Π-stability appears to be correct in this limit as well.

8. Why the grading?

One might consider a simpler definition of stability which only depends on Z/Z' as a conventional complex variable, but this turns out not to be possible. First of all, there cannot be a decay $E \to E' + \ldots$ when Z/Z' is on the negative real axis, by conservation of energy. (The decay $E \to \bar{E}' + \ldots$ might be possible but is covered independently by checking whether $\bar{E}'$ is a subobject of E.) Thus we have no decay when $\varphi(E') - \varphi(E) = 1$.

This argument does not rule out a decay when $\varphi(E') - \varphi(E) = 2$. If this were possible, we would need to keep track of which sheet of the complex plane Z/Z' sits on anyways, to get the direction of decay correct. So there cannot be a condition for stability which depends only on central charges; it must see the grading.

[3] We have only checked these signs carefully for $\mathbb{C}^3/\mathbb{Z}_3$, where they follow if we define the line from the orbifold point to the conifold point to be real $z > 0$.

One should next ask whether the definition of grading in terms of analytically continued central charges is sensible. Indeed, there is no obvious prescription for adding graded central charges, so one cannot analytically continue the periods Π and then define $Z = Q \cdot \Pi$. One can certainly analytically continue all the central charges separately, but one might then expect compatibility conditions between them.

A better conceptual basis for this definition uses the idea of "graded lagrangian submanifold." [18] This idea originated in the work of Fukaya and Kontsevich [19, 20] and has appeared in other discussions of mirror symmetry [21], but it has not played a direct physical role until now.

The basic point is that the symplectic group, $\mathrm{Sp}(2n)$, has a non-zero first homotopy group $\pi_1 \cong \mathbb{Z}$. A general variation of a lagrangian submanifold can be described by an element of $\mathrm{Sp}(2n)$ at each point. Given a metric, we can consider orthonormal frames, and reduce the action to $\mathrm{U}(n)$, its maximal compact subgroup.

For variations of special lagrangian manifolds, one must have the same $\mathrm{U}(1)$ element at each point on the surface, since this acts on the holomorphic three-form. Thus any closed loop, such as one induced from a closed loop in complex moduli space (or Teichmüller space), can be associated with a winding number in $\pi_1(\mathrm{U}(n))$. This allows us to extend the gradings of the sL-manifolds throughout the Teichmüller space.

We still have the problem that there is no clear prescription for adding or subtracting graded central charges. The only obvious way to avoid this problem is to only allow decays to constituents which have the same grade. We have also found in concrete examples that postulating that decays happen for more than one value of $\varphi(E') - \varphi(E)$ generally leads to inconsistencies such as decay to constituents which are more massive than the original object. These two points lead us to rule out such decays in our proposal (2.2).

We should say that, in our opinion, this is the weakest point in the arguments for our proposal. Although our proposal is clearly the simplest of this type, at present it also seems conceivable that the correct condition in string theory is more complicated, with decays at more than one even integral value of $\varphi(E') - \varphi(E)$. Although somewhat problematic, a reason not to rule out this possibility is the idea that $\varphi(E)$ in string theory could be a periodic variable. Although this is not immediately incompatible with (2.2), many further consistency conditions would need to be satisfied. In any case, we expect that further study of the explicit $\mathbb{C}^3/\mathbb{Z}_3$ example will pick out (at most) one viable proposal.

We conclude with the remark that the explicit analyses in the large volume and orbifold limits determine the gradings for the objects under consideration, providing a starting point for the definition by analytic continuation of the central charges. Nontrivial gradings will then arise when one moves large distances in moduli space.

9. Conclusions

D-branes in type-II string compactifications with $\mathcal{N} = 2$ supersymmetry in the bulk are specified by a choice of embedding and a choice of gauge bundle. We proposed a general criterion, Π-stability, for determining the configurations which preserve $\mathcal{N} = 1$ supersymmetry. The condition depends on the moduli of the Calabi-Yau and combines elements of the A and B brane mirror pictures.

In a given example the criterion has two elements, which correspond respectively to the problems of solving "F-flatness" (superpotential) and "D-flatness" conditions in the $\mathcal{N} = 1$ effective theory.

The problem of finding the set of F-flat configurations is independent of the D terms and given our decoupling assumption is independent of the Kähler moduli (although it could happen that in different regimes of Kähler moduli space, very different configurations survive the D-flatness conditions). In the large volume limit these are holomorphic sheaves; more generally one expects a similar category of holomorphic objects which admit a definition of homomorphism. As another example of the category of holomorphic objects, we discussed the orbifold limit of non-compact CY's, in which the category is that of quiver representations. We will discuss the example of $\mathbb{C}^3/\mathbb{Z}_3$ in detail in [1].

The correct category must admit an action by the monodromy transformations of the CY, as discussed in [22, 23]. In principle it can contain constituents which are not simultaneously BPS. The question of what states are simultaneously BPS depends on Kähler moduli, and thus (given the decoupling) it cannot enter in defining the space of holomorphic objects. It is not yet clear whether such a general description can be made with a finite number of degrees of freedom.

Following the seminal proposal of [20], the "derived category" based on the category of sheaves (which in the orbifold example is the same as the derived category based on the quivers) is a natural candidate to explore.[4]

Given these objects, Π-stability is a precise definition of stability of holomorphic objects which can be analyzed just knowing the periods at the point in moduli space of interest, and the inclusion relations between objects. Computing periods is a well-studied problem and obtaining the gradings appears to be easy as well. The inclusion relations are not necessarily easy to get, but they are clearly necessary for analyzing any definition of stability motivated by geometric invariant theory, and seem to carry direct physical information about the products of a decay process. These considerations lead us to believe that this is the simplest general form of the stability condition one could propose. We checked its validity in several limits; conversely, a failure of the condition would appear to contradict one or more elements of the currently accepted picture of branes on CY. (As we noted in section 8, there is a similar but more complicated variant proposal which we have not ruled out at this point.) It is also worth mentioning that there are many other consistency checks one could make with string theory; for example that objects are only destabilized by lighter constituents.

Our proposal was originally motivated by the physics of D-branes in weakly coupled type-II theory, but the concepts required for its statement as well as the principles justifying it are quite general, and one might expect it to apply to BPS states in quite general $\mathcal{N} = 2$

[4](Note added in v3): As pointed out to us by E. Sharpe and by R. Thomas, the derived category does not have a clear notion of subobject, so it is not at all obvious that it can be used as the category of our proposal. In any case we regard it as a useful clue to the correct category, as several of its features do have analogs in string theory [24 – 26].

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theories or at least those which can be embedded in or are dual to D-brane theories in type-II strings. For example, it will be interesting to see if marginal stability in $\mathcal{N} = 2$ supersymmetric gauge theory (as studied for example in [27, 28]) can be described by using a suitable category of objects.

In [1] and subsequent work we will discuss the concrete BPS branes which arise in particular Calabi-Yaus, and analyze their Π-stability. The direct physical applications of such work might include a better understanding of dualities of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories, computations of black hole entropy, and quantitative approaches to the study of supersymmetry breaking.

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